

# REAL ZEROS OF HURWITZ-LERCH ZETA FUNCTIONS IN THE INTERVAL $(-1, 0)$

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**ABSTRACT.** For  $0 < a \leq 1$ ,  $s, z \in \mathbb{C}$  and  $0 < |z| \leq 1$ , the Hurwitz-Lerch zeta function is defined by  $\Phi(s, a, z) := \sum_{n=0}^{\infty} z^n (n+a)^{-s}$  when  $\sigma := \Re(s) > 1$ . In this paper, we show that  $\Phi(\sigma, a, z) \neq 0$  when  $\sigma \in (-1, 0)$  if and only if [I]  $z = 1$  and  $(3 - \sqrt{3})/6 \leq a \leq 1/2$  or  $(3 + \sqrt{3})/6 \leq a \leq 1$ , [II]  $z \in [-1, 1)$  and  $(1-z)(1-a) \leq 1$ , [III]  $z \notin \mathbb{R}$  and  $0 < a \leq 1$ . In addition, we give a new proof of the functional equation of  $\Phi(s, a, z)$ .

## 1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

**1.1. Main results.** The Hurwitz-Lerch zeta function is defined by as follows.

**Definition 1.1** (see [4, p. 53, (1)]). *Let  $0 < a \leq 1$ ,  $s, z \in \mathbb{C}$  and  $0 < |z| \leq 1$ . Then the Hurwitz-Lerch zeta function  $\Phi(s, a, z)$  is defined by*

$$\Phi(s, a, z) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}, \quad s := \sigma + it, \quad \sigma > 1, \quad t \in \mathbb{R}. \quad (1.1)$$

We can easily see that the Riemann zeta function  $\zeta(s)$  and the Hurwitz zeta function  $\zeta(s, a)$  are expressed as  $\Phi(s, 1, 1)$  and  $\Phi(s, a, 1)$ , respectively. The Dirichlet series of  $\Phi(s, a, z)$  converges absolutely in the half-plane  $\sigma > 1$  and uniformly in each compact subset of this half-plane. When  $z \neq 1$ , the function  $\Phi(s, a, z)$  is analytically continuable to the whole complex plane. However, the Hurwitz zeta function  $\zeta(s, a)$  is a meromorphic function with a simple pole at  $s = 1$ . In this paper, we show the following.

**Theorem 1.2.** *Let  $b_2^{\pm} := (3 \pm \sqrt{3})/6$ . Then the Hurwitz-Lerch zeta function  $\Phi(\sigma, a, z)$  does not vanish when  $\sigma \in (-1, 0)$  if and only if*

$$\begin{cases} \text{[I]} & z = 1, & b_2^- \leq a \leq 1/2 \text{ or } b_2^+ \leq a \leq 1, \\ \text{[II]} & z \in [-1, 1), & (1-z)(1-a) \leq 1, \\ \text{[III]} & |z| = 1, z \notin \mathbb{R}, & 0 < a \leq 1. \end{cases}$$

*Note that  $b_2^{\pm}$  are the roots of the second Bernoulli polynomial  $B_2(x) := x^2 - x + 1/6$ .*

In Section 2, we prove Theorem 1.2. During the proof process of Theorem 1.2 (see Proposition 2.4), we show  $\zeta(\sigma, a) > 0$  when  $b_2^- \leq a \leq 1/2$  and  $\zeta(\sigma, a) < 0$  when  $b_2^+ \leq a \leq 1$  in the interval  $(-1, 0)$ . In Section 3, we give a new proof of the functional equations (3.1) and (3.3) by using the integral representations (2.4) and (2.9), and modifying the fifth method of the proof of the functional equation for the Riemann zeta function (see [16, Section 2.8]). It should be noted that in the 21st century, Knopp and Robins [7], and Navas, Ruiz and Varona [9] gave new proofs of the functional equation for  $\zeta(s, a)$  by using Poisson summation of the Lipschitz summation formula (see [7, p. 1916]) and the uniqueness of Fourier coefficients (see [9, p. 190]), respectively.

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**1.2. Known results.** As mentioned in [8, Section 1], the study of real zeros of  $\Phi(s, a, z)$  has a long, long history. Let  $\text{Li}_s(z) := z\Phi(s, 1, z) = \sum_{n=1}^{\infty} z^n n^{-s}$ . Roy [12] proved that  $\text{Li}_\sigma(z) \neq 0$  for all  $|z| \leq 1$ ,  $z \neq 1$  and  $\sigma > 0$ . Berndt [3] showed that  $\zeta(\sigma, a+1) = \zeta(\sigma, a) - a^{-\sigma} \neq 0$  for any  $0 < \sigma < 1$  and  $0 \leq a \leq 1$ .

For zeros of zeta functions in the half-plane  $\sigma < 0$ , we have the following research. Peyerimhoff [11] proved that for (fixed)  $\sigma < 0$ , the function  $\text{Li}_\sigma(z)$  has  $-\lfloor \sigma \rfloor$  simple zeros for  $z \in \mathbb{C} \setminus [1, \infty)$  and they are all non-positive (see also [10, Section 8]). Spira [15] showed that if  $\sigma \leq -4a - 1 - 2[1 - 2a]$  and  $|t| \leq 1$ , then  $\zeta(s, a) \neq 0$  except for zeros on the negative real line, one in each interval  $(-2n - 4a - 1, -2n - 4a + 1)$ ,  $\mathbb{N} \ni n \geq 1 - 2a$ . Some similar results for the Lerch zeta function  $\Phi(s, a, e^{2\pi i \theta})$  with  $0 < \theta \leq 1$  are given by Garunkštis and Laurinćikas [5]. Veselov and Ward [17] proved that  $\zeta(-\sigma, a)$  has no real zeros in the region  $4\pi ea > 1 + 2\sigma + \log \sigma$  for large  $\sigma$ .

For real zeros of the Hurwitz zeta function in the interval  $(0, 1)$ , Schipani [13] showed that  $\zeta(\sigma, a)$  has no zeros when  $0 < \sigma < 1$  and  $1 - \sigma \leq a$ . The author [8] proved that  $\zeta(\sigma, a)$  does not vanish for all  $0 < \sigma < 1$  if and only if  $a \geq 1/2$ , and  $\Phi(\sigma, a, z) \neq 0$  for all  $0 < \sigma < 1$  and  $0 < a \leq 1$  when  $z \neq 1$ .

**1.3. Some remarks.** The Dirichlet  $L$ -function with a Dirichlet character  $\chi$  is defined by  $L(s, \chi) := \sum_{n=1}^{\infty} \chi(n)n^{-s}$ . Let  $\varphi$  be the Euler totient function and  $\chi$  be a primitive Dirichlet character of conductor of  $q$ . Then the following six relations between  $L(s, \chi)$ ,  $\zeta(s, a)$  and  $\text{Li}_s(z)$  are well-known;

$$\begin{aligned} L(s, \chi) &= q^{-s} \sum_{r=1}^q \chi(r) \zeta(s, r/q), & \zeta(s, r/q) &= \frac{q^s}{\varphi(q)} \sum_{\chi \bmod q} \bar{\chi}(r) L(s, \chi), \\ \zeta(s, r/q) &= \sum_{n=1}^q e^{-2\pi i r n/q} \text{Li}_s(e^{2\pi i r n/q}), & \text{Li}_s(e^{2\pi i r/q}) &= q^{-s} \sum_{n=1}^q e^{2\pi i r n/q} \zeta(s, n/q), \\ L(s, \chi) &= \frac{1}{G(\bar{\chi})} \sum_{r=1}^q \bar{\chi}(r) \text{Li}_s(e^{2\pi i r/q}), & \text{Li}_s(e^{2\pi i r/q}) &= \frac{1}{\varphi(q)} \sum_{\chi \bmod q} G(\bar{\chi}) L(s, \chi), \end{aligned}$$

where  $G(\bar{\chi}) := \sum_{n=1}^q \bar{\chi}(n) e^{2\pi i r n/q}$  denotes the Gauss sum associated to  $\bar{\chi}$ .

Siegel [14] proved that for any  $\varepsilon > 0$ , there exists  $C_\varepsilon > 0$  such that, if  $\chi$  is a real primitive Dirichlet character modulo  $q$ , then  $L(1, \chi) > C_\varepsilon q^{-\varepsilon}$ . It is considered likely that  $L(\sigma, \chi) \neq 0$  for all  $0 < \sigma < 1$ , namely, so-called Siegel zeros do not exist. From the Euler product and the functional equation of the Dirichlet  $L$ -function,  $L(\sigma, \chi)$  does not vanish for  $\sigma \in (-1, 0)$ . This fact should be compared with the following deduced by Theorem 1.2 that  $\zeta(\sigma, a) \neq 0$  for all  $\sigma \in (-1, 0)$  if and only if  $(3 - \sqrt{3})/6 \leq a \leq 1/2$  or  $(3 + \sqrt{3})/6 \leq a \leq 1$  and  $\text{Li}_\sigma(z) \neq 0$  for all  $z \neq 1$ ,  $|z| = 1$  and  $\sigma \in (-1, 0)$ .

Taking known results and Theorem 1.2 altogether, we have the following.

	$-1 < \sigma < 0$	$0 < \sigma < 1$	$1 < \sigma$
$\zeta(\sigma, a) \neq 0$	$b_2^- \leq a \leq 1/2, b_2^+ \leq a \leq 1$	$a \geq 1/2$	$0 < a \leq 1$
$\text{Li}_\sigma(z) \neq 0, z \neq 1$	$z \neq 1$	$z \neq 1$	$z \neq 1$
$L(\sigma, \chi) \neq 0, \text{ real } \chi$	$\text{real } \chi$	$\text{Siegel zero}$	$\text{real } \chi$

## 2. PROOF OF THEOREM 1.2

**2.1. Proof of Theorem 1.2 (I).** To prove (I) of Theorem 1.2, we define  $H(a, x)$  by

$$H(a, x) := \frac{e^{(1-a)x}}{e^x - 1} - \frac{1}{x} = \frac{xe^{(1-a)x} - e^x + 1}{x(e^x - 1)}, \quad x > 0. \quad (2.1)$$

We have to mention that for  $|x| \leq 1$ ,

$$H(a, x) = \frac{1}{x} \sum_{n=1}^{\infty} B_n(1-a) \frac{x^n}{n!} = \frac{1}{2} - a + \frac{B_2(a)}{2!}x + \cdots, \quad (2.2)$$

where  $B_n(a)$  is the  $n$ -th Bernoulli polynomial (see for example [2, p. 246] or [1, Proposition 4.9]). We quote the following lemma from [8].

**Lemma 2.1** (see [8, Lemma 2.1]). *For  $0 < \sigma < 1$ ,*

$$\Gamma(s)\zeta(s, a) = \int_0^{\infty} \left( \frac{e^{(1-a)x}}{e^x - 1} - \frac{1}{x} \right) x^{s-1} dx. \quad (2.3)$$

When  $-1 < \sigma < 0$ , we have the following integral representation.

**Proposition 2.2** (see [9, (2.4)]). *For  $-1 < \sigma < 0$ , it holds that*

$$\Gamma(s)\zeta(s, a) = \int_0^{\infty} \left( \frac{e^{(1-a)x}}{e^x - 1} - \frac{1}{x} - \frac{1}{2} + a \right) x^{s-1} dx. \quad (2.4)$$

*Proof.* For reader's convenience, we give a proof here. When  $0 < \sigma < 1$ , we have

$$\Gamma(s)\zeta(s, a) = \int_0^{\infty} H(a, x) x^{s-1} dx$$

by (2.1) and Lemma 2.1. Hence we have

$$\begin{aligned} \Gamma(s)\zeta(s, a) &= \int_0^1 H(a, x) x^{s-1} dx + \int_1^{\infty} H(a, x) x^{s-1} dx \\ &= \int_0^1 \left( H(a, x) - \frac{1}{2} + a \right) x^{s-1} dx + \left( \frac{1}{2} - a \right) \int_0^1 x^{s-1} dx + \int_1^{\infty} H(a, x) x^{s-1} dx \\ &= \int_0^1 G(a, x) x^{s-1} dx + \int_1^{\infty} H(a, x) x^{s-1} dx + \left( \frac{1}{2} - a \right) \frac{1}{s}, \end{aligned}$$

where the function  $G(a, x)$  is defined as

$$G(a, x) := \frac{e^{(1-a)x}}{e^x - 1} - \frac{1}{x} - \frac{1}{2} + a = H(a, x) - \left( \frac{1}{2} - a \right). \quad (2.5)$$

For  $\sigma > -1$ , it holds that

$$\int_0^1 |G(a, x)| |x^{s-1}| dx \ll \int_0^1 x^{\sigma} dx = \frac{1}{1+\sigma} < \infty$$

by (2.2) and (2.5). On the other hand, we have

$$\frac{1}{s} = - \int_1^{\infty} x^{s-1} dx, \quad -1 < \sigma < 0. \quad (2.6)$$

Furthermore, when  $-1 < \sigma < 0$  one has

$$\int_1^{\infty} |H(a, x)| |x^{s-1}| dx \ll \int_1^{\infty} \frac{e^{(1-a)x}}{e^x - 1} x^{\sigma-1} dx < \infty, \quad \int_1^{\infty} |x^{s-1}| dx = \int_1^{\infty} x^{\sigma-1} dx < \infty.$$

Therefore, the integral representation

$$\Gamma(s)\zeta(s, a) = \int_0^1 G(a, x) x^{s-1} dx + \int_1^{\infty} H(a, x) x^{s-1} dx - \left( \frac{1}{2} - a \right) \int_1^{\infty} x^{s-1} dx$$

gives an analytic continuation for  $-1 < \sigma < 0$ . Hence we obtain this Proposition.  $\square$

**Lemma 2.3.** *The function  $G(a, x)$  defined by (2.5) is negative for all  $x > 0$  if and only if  $b_2^- := (3 - \sqrt{3})/6 \leq a \leq 1/2$ . Moreover, we have  $G(a, x) > 0$  for all  $x > 0$  if and only if  $a \geq (3 + \sqrt{3})/6 =: b_2^+$ .*

*Proof.* Obviously, one has

$$G(a, x) = \frac{xe^{(1-a)x} - e^x + 1 - (1/2 - a)x(e^x - 1)}{x(e^x - 1)}.$$

Since  $x(e^x - 1) > 0$  for all  $x > 0$ , we consider the numerator of  $G(a, x)$  written as

$$g(a, x) := x(e^x - 1)G(a, x) = xe^{(1-a)x} - e^x + 1 - (1/2 - a)x(e^x - 1).$$

By differentiating with respect to  $x$ , we have

$$\begin{aligned} g'(a, x) &= (1 - a)xe^{(1-a)x} + e^{(1-a)x} - e^x - (1/2 - a)(xe^x + e^x - 1), \\ g''(a, x) &= ((1 - a)^2x + 2(1 - a))e^{(1-a)x} - e^x - (1/2 - a)(xe^x + 2e^x). \end{aligned}$$

Obviously, one has

$$g(a, 0) = g'(a, 0) = g''(a, 0) = 0. \quad (2.7)$$

Now we consider the function

$$e^{(a-1)x}g''(a, x) = (1 - a)^2x + 2(1 - a) - (1/2 - a)xe^{ax} - 2(1 - a)e^{ax}.$$

First suppose  $0 < a \leq 1/2$ . Then we have  $\lim_{x \rightarrow \infty} e^{(a-1)x}g''(a, x) = -\infty$ . By the Taylor expansion of  $e^{ax} = \sum_{n=0}^{\infty} (ax)^n/n!$ , one has

$$\begin{aligned} (1/2 - a)xe^{ax} + 2(1 - a)e^{ax} &= (1/2 - a)(x + ax^2 + \cdots) + 2(1 - a)(1 + ax + \cdots) \\ &= 2(1 - a) + ((1/2 - a) + 2a(1 - a))x + \cdots. \end{aligned}$$

In this case, we have

$$(1 - a)^2x + 2(1 - a) < (1/2 - a)xe^{ax} + 2(1 - a)e^{ax}, \quad x > 0$$

if  $(1 - a)^2 \leq (1/2 - a) + 2a(1 - a)$  which is equivalent to  $3B_2(a) = 3a^2 - 3a + 1/2 \leq 0$ , where  $B_2(a)$  is the second Bernoulli polynomial. Hence one has  $g''(a, x) < 0$  for all  $x > 0$  when  $(3 - \sqrt{3})/6 \leq a \leq 1/2$ . Thus we obtain  $g'(a, x) < 0$  and  $g(a, x) < 0$  for all  $x > 0$  when  $b_2^- \leq a \leq 1/2$  from (2.7).

Next assume  $a \geq 1/2$ . Then one has  $\lim_{x \rightarrow \infty} e^{(a-1)x}g''(a, x) = \infty$ . By the Taylor expansion of  $e^{ax} = \sum_{n=0}^{\infty} (ax)^n/n!$ , we have

$$(a - 1/2)xe^{ax} = \sum_{n=1}^{\infty} \frac{(a - 1/2)a^{n-1}}{(n-1)!}x^n, \quad 2(1 - a)e^{ax} = \sum_{n=0}^{\infty} \frac{2(1 - a)a^n}{n!}x^n.$$

For  $n \geq 2$  and  $a \geq 3/4$ , it holds that

$$\frac{(a - 1/2)a^{n-1}}{(n-1)!} \geq \frac{2(1 - a)a^n}{n!}$$

since we have

$$n(a - 1/2) \geq 2(a - 1/2) \geq 1/2 \geq -2(a - 1/2)^2 + 1/2 = 2a(1 - a).$$

Note that  $(3 + \sqrt{3})/6 = 0.788675... > 3/4$ . Thus we have

$$(1 - a)^2x + 2(1 - a) + (a - 1/2)xe^{ax} > 2(1 - a)e^{ax}, \quad x > 0$$

if  $(1 - a)^2 + (a - 1/2) \geq 2a(1 - a)$  which is equivalent to  $3B_2(a) = 3a^2 - 3a + 1/2 \geq 0$ . Thus we obtain  $g''(a, x) > 0$  for all  $x > 0$  when  $a \geq (3 + \sqrt{3})/6$ . Hence one has  $g'(a, x) > 0$  and  $g(a, x) > 0$  for all  $x > 0$  when  $a \geq b_2^+$  from (2.7).

Finally suppose  $0 < a < b_2^-$  or  $1/2 < a < b_2^+$ . By (2.2) and (2.5), one has

$$2G(a, x) = B_2(a)x + O(x^2).$$

Hence we have

$$G(a, x) > 0, \quad 0 < a < b_2^- \quad \text{and} \quad G(a, x) < 0, \quad 1/2 < a < b_2^+$$

when  $x > 0$  is sufficiently small. On the other hand, it holds that

$$G(a, x) < 0, \quad 0 < a < b_2^- \quad \text{and} \quad G(a, x) > 0, \quad 1/2 < a < b_2^+$$

when  $x > 0$  is sufficiently large from the term  $(1/2 - a)x(e^x - 1)$  in  $G(a, x)$ . Hence we have this lemma.  $\square$

By using Proposition 2.2, Lemma 2.3 and the fact that  $\Gamma(\sigma) < 0$  when  $-1 < \sigma < 0$ , we have the following.

**Proposition 2.4.** *For  $-1 < \sigma < 0$ , one has*

$$\zeta(\sigma, a) > 0, \quad b_2^- \leq a \leq 1/2 \quad \text{and} \quad \zeta(\sigma, a) < 0, \quad b_2^+ \leq a \leq 1.$$

*Proof of Theorem 1.2 (I).* We only have to consider the cases  $0 < a < b_2^-$  or  $1/2 < a < b_2^+$  from Proposition 2.4. It is well-known that

$$\zeta(0, a) = \frac{1}{2} - a, \quad \zeta(-1, a) = -\frac{a^2}{2} + \frac{a}{2} - \frac{1}{12} = -\frac{B_2(a)}{2}$$

(see [1, Proposition 9.3] or [2, Theorem 12.3]). Thus it holds that

$$\begin{aligned} \zeta(0, a) > 0 \quad \text{and} \quad \zeta(-1, a) < 0, & \quad \text{when} \quad 0 < a < b_2^-, \\ \zeta(0, a) < 0 \quad \text{and} \quad \zeta(-1, a) > 0, & \quad \text{when} \quad 1/2 < a < b_2^+. \end{aligned}$$

Therefore, we have Theorem 1.2 (I).  $\square$

**2.2. Proof of Theorem 1.2 (II) and (III).** We quote the following integral representation of Hurwitz-Lerch zeta function  $\Phi(s, a, z)$ .

**Lemma 2.5** (see [4, p. 53, (3)]). *When  $z \neq 1$ , we have*

$$\Phi(s, a, z) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1} e^{(1-a)x}}{e^x - z} dx, \quad \Re(s) > 0. \quad (2.8)$$

By using the integral formula above, we obtain the following.

**Proposition 2.6.** *For  $z \neq 1$ , one has*

$$\Phi(s, a, z) = \frac{1}{\Gamma(s)} \int_0^\infty \left( \frac{e^{(1-a)x}}{e^x - z} - \frac{1}{1 - z} \right) x^{s-1} dx, \quad -1 < \Re(s) < 0. \quad (2.9)$$

*Proof.* When  $\sigma > 0$ , we have

$$\begin{aligned} \Gamma(s)\Phi(s, a, z) &= \int_0^1 \frac{x^{s-1} e^{(1-a)x}}{e^x - z} dx + \int_1^\infty \frac{x^{s-1} e^{(1-a)x}}{e^x - z} dx \\ &= \int_0^1 \left( \frac{e^{(1-a)x}}{e^x - z} - \frac{1}{1 - z} \right) x^{s-1} dx + \int_0^1 \frac{x^{s-1}}{1 - z} dx + \int_1^\infty \frac{x^{s-1} e^{(1-a)x}}{e^x - z} dx \\ &= \int_0^1 \left( \frac{e^{(1-a)x}}{e^x - z} - \frac{1}{1 - z} \right) x^{s-1} dx + \int_1^\infty \frac{x^{s-1} e^{(1-a)x}}{e^x - z} dx + \frac{1}{s(1 - z)} \end{aligned}$$

by using (2.8). From

$$\lim_{x \rightarrow 0+} \left( \frac{e^{(1-a)x}}{e^x - z} - \frac{1}{1 - z} \right) = 0,$$

we can see that

$$\int_0^1 \left| \frac{e^{(1-a)x}}{e^x - z} - \frac{1}{1-z} \right| |x^{s-1}| dx \ll \int_0^1 x^\sigma dx < \infty$$

when  $-1 < \Re(s) < 0$ . Hence we have this proposition by (2.6).  $\square$

In order to prove Theorem 1.2 (II), we show the following.

**Proposition 2.7.** *When  $z \in [-1, 1)$  and  $(1-a)(1-z) \leq 1$ , we have*

$$\Phi(\sigma, a, z) > 0, \quad -1 < \sigma < 0.$$

*Proof.* Let  $z \in [-1, 1)$  and put

$$G_z(a, x) := \frac{e^{(1-a)x}}{e^x - z} - \frac{1}{1-z} = \frac{(1-z)e^{(1-a)x} - e^x + z}{(1-z)(e^x - z)}.$$

Hence consider the numerator of  $G_z(a, x)$  expressed as

$$g_z(a, x) := (1-z)e^{(1-a)x} - e^x + z.$$

Then we obtain  $g_z(a, 0) = 0$  and

$$g'_z(a, x) = (1-z)(1-a)e^{(1-a)x} - e^x.$$

Obviously  $g'_z(a, x) < 0$  is equivalent to  $(1-z)(1-a) < e^{ax}$ . We can see that  $(1-z)(1-a) < e^{ax}$  for all  $x > 0$  if  $(1-a)(1-z) \leq 1$ . Hence, we have this proposition by (2.9) and the fact that  $\Gamma(\sigma) < 0$  when  $-1 < \sigma < 0$ .  $\square$

*Proof of Theorem 1.2 (II).* We only have to prove that  $\Phi(\sigma, a, z)$  has zeros in the interval  $(-1, 0)$  when  $(1-a)(1-z) > 1$ . From [6, p. 17], one has

$$\Phi(0, a, z) = \frac{1}{1-z} > 0, \quad \Phi(-1, a, z) = \frac{a}{1-z} + \frac{z}{(1-z)^2}.$$

When  $(1-a)(1-z) > 1$ , we have  $\Phi(-1, a, z) < 0$ . Therefore,  $\Phi(\sigma, a, z)$  vanishes in the interval  $(-1, 0)$  if  $(1-a)(1-z) > 1$  by the intermediate value theorem.  $\square$

*Proof of Theorem 1.2 (III).* For  $0 < r \leq 1$ ,  $0 < \theta < \pi$  and  $\pi < \theta < 2\pi$ , put

$$\begin{aligned} G_{r,\theta}(a, x) &:= \frac{e^{(1-a)x}}{e^x - re^{i\theta}} - \frac{1}{1-re^{i\theta}} \\ &= \frac{e^{(1-a)x}((e^x - r \cos \theta) + ir \sin \theta)}{(e^x - r \cos \theta)^2 + r^2 \sin^2 \theta} - \frac{(1 - r \cos \theta) + ir \sin \theta}{(1 - r \cos \theta)^2 + r^2 \sin^2 \theta}. \end{aligned}$$

Obviously, we obtain that

$$\Im(G_{r,\theta}(a, x)) = \frac{e^{(1-a)x} r \sin \theta}{e^{2x} + r^2 - 2e^x r \cos \theta} - \frac{r \sin \theta}{1 + r^2 - 2r \cos \theta}.$$

Now we consider the following three functions

$$\begin{aligned} g_{r,\theta}^b(a, x) &:= e^{(1-a)x} (1 + r^2 - 2r \cos \theta), & g_{r,\theta}^\#(a, x) &:= e^{2x} + r^2 - 2e^x r \cos \theta, \\ g_{r,\theta}^{\natural}(a, x) &:= e^{2(1-a)x} + r^2 - 2e^{(1-a)x} r \cos \theta. \end{aligned}$$

For all  $x > 0$ , we can show  $g_{r,\theta}^b(a, x) < g_{r,\theta}^{\natural}(a, x) < g_{r,\theta}^\#(a, x)$  from

$$\begin{aligned} g_{r,\theta}^\#(a, x) - g_{r,\theta}^{\natural}(a, x) &= (e^x - e^{(1-a)x})(e^x + e^{(1-a)x} - 2r \cos \theta) > 0, \\ g_{r,\theta}^{\natural}(a, x) - g_{r,\theta}^b(a, x) &= (e^{(1-a)x} - 1)(e^{(1-a)x} - r^2) > 0. \end{aligned}$$

Hence  $\Im(G_{r,\theta}(a, x)/\sin \theta) < 0$  for all  $x > 0$ . Therefore, we obtain  $\Im(\Phi(\sigma, a, z)) \neq 0$  when  $-1 < \Re(s) < 0$  for  $z = re^{i\theta}$ , where  $0 < r \leq 1$ ,  $0 < \theta < \pi$  and  $\pi < \theta < 2\pi$ .  $\square$

## 3. NEW PROOFS OF THE FUNCTIONAL EQUATIONS

3.1. **The case  $z = 1$ .** In this subsection, we show the functional equation

$$\zeta(s, a) = \frac{(-\pi i)(2\pi)^{s-1}}{\Gamma(s) \sin \pi s} \left( e^{\pi i s/2} \sum_{n=1}^{\infty} \frac{e^{2\pi i n a}}{n^{1-s}} - e^{-\pi i s/2} \sum_{n=1}^{\infty} \frac{e^{-2\pi i n a}}{n^{1-s}} \right), \quad (3.1)$$

where  $0 < a < 1$  and  $\Re(s) < 0$  (see for instance [1, Theorem 9.4]) by the integral representation (2.4). The following formulas are well-known.

**Lemma 3.1** (see the proof of [1, Theorem 4.11]). *For  $0 < a < 1$ , one has*

$$\frac{e^{ax}}{e^x - 1} - \frac{1}{x} = \lim_{N \rightarrow \infty} \sum_{0 \neq n=N}^{-N} \frac{e^{2\pi i n a}}{x - 2\pi i n}, \quad a - \frac{1}{2} = \lim_{N \rightarrow \infty} \sum_{0 \neq n=N}^{-N} \frac{e^{2\pi i n a}}{-2\pi i n}.$$

Recall  $G(a, x)$  is defined by (2.5). From the lemma above, we have

$$\begin{aligned} G(a, x) &= \lim_{N \rightarrow \infty} \sum_{0 \neq n=N}^{-N} \left( \frac{e^{-2\pi i n a}}{x - 2\pi i n} - \frac{e^{2\pi i n a}}{2\pi i n} \right) \\ &= \sum_{n=1}^{\infty} \frac{x e^{-2\pi i n a}}{2\pi i n (x - 2\pi i n)} - \sum_{n=1}^{\infty} \frac{x e^{2\pi i n a}}{2\pi i n (x + 2\pi i n)}. \end{aligned} \quad (3.2)$$

**Lemma 3.2.** *When  $-1 < \Re(s) < 0$ , we have*

$$\int_0^{\infty} \frac{x^s dx}{2\pi i n (x - 2\pi i n)} = \frac{(2\pi)^s e^{\pi i s/2}}{1 - e^{2\pi i s}} n^{s-1}, \quad \int_0^{\infty} \frac{x^s dx}{2\pi i n (x + 2\pi i n)} = \frac{(2\pi)^s e^{3\pi i s/2}}{1 - e^{2\pi i s}} n^{s-1}.$$

*Proof.* Let  $R > r > 0$ . Then consider the following contour integral,

$C_1$  : the part of real axis from  $r$  to  $R$ ,

$I(R)$  : the circle of radius  $R$  with center at the origin (oriented counter-clockwise),

$C_2$  : the part of real axis from  $R$  to  $r$ ,

$I(r)$  : the circle of radius  $r$  with center at the origin (oriented clockwise).

Then we have  $\int_{I(R)} \rightarrow 0$  when  $R \rightarrow \infty$  and  $\int_{I(r)} \rightarrow 0$  when  $r \rightarrow 0$ . Hence it holds that

$$\begin{aligned} (1 - e^{2\pi i s}) \int_0^{\infty} \frac{x^s dx}{2\pi i n (x - 2\pi i n)} &= 2\pi i \frac{(2\pi i n)^s}{2\pi i n} = (2\pi)^s e^{\pi i s/2} n^{s-1}, \\ (1 - e^{2\pi i s}) \int_0^{\infty} \frac{x^s dx}{2\pi i n (x + 2\pi i n)} &= 2\pi i \frac{(-2\pi i n)^s}{2\pi i n} = (2\pi)^s e^{3\pi i s/2} n^{s-1} \end{aligned}$$

by the residue theorem. The formulas above imply Lemma 3.2.  $\square$

*Proof of (3.1).* For  $0 < a < 1$  and  $-1 < \Re(s) < 0$ , one has

$$\begin{aligned} \Gamma(s) \zeta(s, a) &= \int_0^{\infty} G(a, x) x^{s-1} dx = \int_0^{\infty} \sum_{n=1}^{\infty} \frac{e^{-2\pi i n a} x^s dx}{2\pi i n (x - 2\pi i n)} - \int_0^{\infty} \sum_{n=1}^{\infty} \frac{e^{2\pi i n a} x^s dx}{2\pi i n (x + 2\pi i n)} \\ &= \frac{2\pi i (2\pi)^{s-1} e^{\pi i s/2}}{e^{2\pi i s} - 1} \sum_{n=1}^{\infty} \frac{e^{-2\pi i n a}}{n^{1-s}} - \frac{2\pi i (2\pi)^{s-1} e^{3\pi i s/2}}{e^{2\pi i s} - 1} \sum_{n=1}^{\infty} \frac{e^{2\pi i n a}}{n^{1-s}} \\ &= \frac{\pi i (2\pi)^{s-1} e^{-\pi i s/2}}{\sin \pi s} \sum_{n=1}^{\infty} \frac{e^{-2\pi i n a}}{n^{1-s}} - \frac{\pi i (2\pi)^{s-1} e^{\pi i s/2}}{\sin \pi s} \sum_{n=1}^{\infty} \frac{e^{2\pi i n a}}{n^{1-s}}. \end{aligned}$$

by (3.2), Proposition 2.2 and Lemma 3.2. The interchange the order of the integral and the summation are justified by Lebesgue's dominated convergence theorem.  $\square$

**3.2. The case  $z \neq 1$ .** Next, we prove the functional equation

$$\Phi(s, a, z) = z^{-a} \Gamma(1-s) \sum_{n=-\infty}^{\infty} (-\log z + 2\pi in)^{s-1} e^{2\pi ina}, \quad (3.3)$$

where  $0 < a < 1$ ,  $z \neq 1$  and  $\Re(s) < 0$  (see [4, p. 28, (6)]) by the using integral representation (2.9). The following is a generalization of Lemma 3.2.

**Lemma 3.3.** *For  $0 < a < 1$  and  $z \neq 1$ , we have*

$$\frac{e^{(1-a)x}}{e^x - z} = \lim_{N \rightarrow \infty} \sum_{n=N}^{-N} \frac{z^{-a} e^{-2\pi ina}}{x - 2\pi in - \log z}, \quad \frac{-1}{1-z} = \lim_{N \rightarrow \infty} \sum_{n=N}^{-N} \frac{z^{-a} e^{-2\pi ina}}{2\pi in + \log z}.$$

*Proof.* The function  $e^{(1-a)y}(e^y - z)^{-1}$  has poles at  $y = 2\pi in + \log z$  with  $n \in \mathbb{Z}$  and all of them are of order 1. The residue at  $y = 2\pi in + \log z$  is expressed as

$$\lim_{y \rightarrow 2\pi in + \log z} (y - 2\pi in - \log z) \frac{e^{(1-a)y}}{e^y - z} = \frac{e^{(1-a)(2\pi in + \log z)}}{z} = z^{-a} e^{-2\pi ina}.$$

Let  $N$  be a sufficiently large natural number and  $C_N$  be a square path passing through four corner points  $R + iR$ ,  $-R + iR$ ,  $-R - iR$  and  $R - iR$ , where  $R = 2\pi(N + 1/2)$ , in this order. If  $x$  is a point inside  $C_N$  such that  $x \neq 2\pi in + \log z$ , one has

$$\int_{C_N} \frac{e^{(1-a)y}}{e^y - z} \frac{dy}{y - x} = 2\pi i \left( \frac{e^{(1-a)x}}{e^x - z} - \sum_{n=N}^{-N} \frac{z^{-a} e^{-2\pi ina}}{x - 2\pi in - \log z} \right)$$

from the residue theorem. When  $N \rightarrow \infty$ , the left hand side tends to 0. Hence we have the first equation of this lemma (see also the proof of [1, Theorem 4.11]). We obtain the second formula of Lemma 3.3 by taking  $x \rightarrow 0$ .  $\square$

From the lemma above, it holds that

$$\begin{aligned} \frac{e^{(1-a)x}}{e^x - z} - \frac{1}{1-z} &= \lim_{N \rightarrow \infty} \sum_{n=N}^{-N} \left( \frac{z^{-a} e^{-2\pi ina}}{x - 2\pi in - \log z} + \frac{z^{-a} e^{-2\pi ina}}{2\pi in + \log z} \right) \\ &= \sum_{n=-\infty}^{\infty} \frac{x z^{-a} e^{-2\pi ina}}{(2\pi in + \log z)(x - 2\pi in - \log z)}. \end{aligned} \quad (3.4)$$

Furthermore, we have

$$\int_0^\infty \frac{x^s dx}{x - 2\pi in - \log z} = \frac{2\pi i}{1 - e^{2\pi is}} (2\pi in + \log z)^s \quad (3.5)$$

by modifying the proof of Lemma 3.2.

*Proof of (3.3).* When  $0 < a < 1$ ,  $z \neq 1$  and  $-1 < \Re(s) < 0$ , we have

$$\begin{aligned} \Phi(s, a, z) &= \frac{1}{\Gamma(s)} \int_0^\infty \left( \frac{e^{(1-a)x}}{e^x - z} - \frac{1}{1-z} \right) x^{s-1} dx \\ &= \frac{1}{\Gamma(s)} \int_0^\infty \sum_{n=-\infty}^{\infty} \frac{x z^{-a} e^{-2\pi ina} dx}{(2\pi in + \log z)(x - 2\pi in - \log z)} = \frac{1}{\Gamma(s)} \frac{2\pi i}{e^{2\pi is} - 1} \sum_{n=-\infty}^{\infty} \frac{-z^{-a} e^{-2\pi ina}}{(2\pi in + \log z)^{1-s}} \\ &= \frac{\Gamma(1-s)}{e^{\pi is}} \sum_{n=-\infty}^{\infty} \frac{-z^{-a} e^{2\pi ina}}{(-2\pi in + \log z)^{1-s}} = z^{-a} \Gamma(1-s) \sum_{n=-\infty}^{\infty} \frac{e^{2\pi ina}}{(2\pi in - \log z)^{1-s}} \end{aligned}$$

from (2.9), (3.4) and (3.5). Thus we have (3.3).  $\square$



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